A VOLUME-AREA INEQUALITY

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ABSTRACT

We prove a Volume-Area Inequality for quasiregular mappings which extends the familiar Ahlfors-Cartwright Length-Area Inequality for analytic functions. Using the Volume-Area Inequality, we derive a distortion theorem, and some boundary behaviour for a special case, when the mappings are mean p-valent in volume with an empty branch set.

1. *Notation and terminology*

Notation and terminology are in general as in [5]; in particular for $x \in \mathbb{R}^n$ we write $x = \sum_{i=1}^{n} x_i e_i$ where $e_1, \ldots, e_n$ are the coordinate unit vectors in $\mathbb{R}^n$. For $a \in \mathbb{R}^n$ and $r > 0$, we write $B^n(a, r) = \{x \in \mathbb{R}^n : |x-a| < r\}$, $B^n(r) = B^n(0, r)$, $B^n = B^n(1)$, $S^{n-1}(a, r) = \partial B^n(a, r)$, $S^{n-1}(r) = \partial B^n(r)$ and $S^{n-1} = \partial B^n$.

The Lebesgue measure of a set $A \subset \mathbb{R}^n$ is written as $m_n(A)$; $\omega_{n-1} = m_{n-1}(S^{n-1})$, $\Omega_n = m_n(B^n)$.

We denote by $\mathbb{R}^{(n-1)}$ the $(n-1)$-plane, $\mathbb{R}^{(n-1)} = \{x \in \mathbb{R}^n : x_n = 0\}$.

For p a positive integer, $A \subset \mathbb{R}^n$, and $t > 0$ we write

$$H^p\{A, t\} = \inf \left\{ \sum_j \Omega_p 2^{-p} \operatorname{dia}(A_j)^p \right\}$$

where the infimum is taken over all countable coverings of A by sets A_j with $\operatorname{dia}(A_j) < t$. The normalized p-dimensional Hausdorff outer measure of A is defined as

$$H^p\{A\} = \lim_{t \to 0} H^p\{A, t\}. \tag{1.1}$$

The closure $\bar{A}$, the boundary ∂A and the complement CA of a set A in $\mathbb{R}^n$ are taken with respect to $\mathbb{R}^n$.

When writing $f: D \to \mathbb{R}^n$, we assume throughout that D is a domain in $\mathbb{R}^n$, f is continuous , and $n \geq 2$.

If $A \subset D$, $y \in \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, we define the following multiplicity (possibly infinite) functions:

$$N(y, f, A) = \operatorname{card}\{f^{-1}(y) \cap A\},$$

$$N(B, f, A) = \sup_{y \in B} N(y, f, A),$$

$$N(f, A) = N(\mathbb{R}^n, f, A),$$

$$N(f) = N(\mathbb{R}^n, f, D).$$

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If $f: D \to \mathbb{R}^n$ is sense-preserving, discrete and open, every $x \in D$ has arbitrarily small normal neighbourhoods U (that is, domains U with $\bar{U} \subset D$, $f(\partial(U)) = \partial(f(U))$ and $U \cap f^{-1}(f(x)) = \{x\}$) with connected complement in $\mathbb{R}^n$ [5; 2.9].

The local topological index of f at a point $x \in D$, denoted by $i(x,f)$, may be defined as

$$i(x,f) = N(f, U) \tag{1.2}$$

where U is any normal neighbourhood for x [5; Theorem 2.12]. Finally, we define

$$n(y,f, D) = \sum_{x \in f^{-1}(y)} i(x,f)\,. \tag{1.3}$$

The branch set, denoted by B_f, is the set of points where f fails to be a local homeomorphism at any neighbourhood of those points.

2. *Quasiregular mappings*

A mapping $f: D \to \mathbb{R}^n$ is said to be quasiregular (qr) if

(i) f is ACL^n (that is, f is locally absolutely continuous on almost all line segments parallel to the coordinate axes, and its partial derivatives belong to $L^n_{\mathrm{loc}}(D)$);

(ii) there exists a constant $K \geqslant 1$ such that

$$|f'(x)|^n \leqslant KJ(x,f) \quad \text{a.e. in } D\,.$$

Here $f' = \left(\dfrac{\partial f_i}{\partial x_j}\right)^n_{i,j=1}$ is the formal derivative of f, $|f'(x)|$ denotes the supremum norm of the linear operator $f'(x)$ and $J(x,f) = \det f'(x)$. A mapping $f: D \to \mathbb{R}^n$ is said to be quasiconformal (qc) if f is qr and injective. We denote by $K_I(f)$, $K_O(f)$ and $K(f)$, respectively, the inner, outer, and maximal dilatation of f, and $l(f'(x)) = \inf_{|h|=1} |f'(x) \cdot h|$; see [5].

3. *A volume-area inequality*

In this section we present an inequality which relates the n-dimensional measure of a domain D of a quasiregular mapping and the $(n-1)$ dimensional measure of certain subsets of D.

LEMMA 3.1. *Let $f: D = \mathbb{R}^n$, $n \geqslant 2$, be a K-quasiregular mapping. Then*

(i) *$|f|$ is absolutely continuous in the sense of Tonelli on D (for the definition see* [11; 2.2]*); or equivalently $|f| \in \mathrm{ACL}^1$;*

(ii) *for every measurable set A such that $m_n(A) = 0$,*

$$H^{n-1}\{x \in A : |f(x)| = R\} = 0\,,$$

for almost every $R > 0$. *Especially when* $A = B_f$,

$$H^{n-1}\{x \in B_f : |f(x)| = R\} = 0,$$

for almost every $R > 0$.

Proof. (i) This follows immediately from the definitions.

(ii) By [**11**; Theorem 2.2.1], as $|f| \in \mathrm{ACL}^1$ we get

$$\int_A |\nabla |f|| dm_n = \int_0^\infty H^{n-1}\{|f|^{-1}(R) \cap A\} dm_1(R).$$

As $m_n(A) = 0$ we get that

$$\int_0^\infty H^{n-1}\{|f|^{-1}(R) \cap A\} dm_1(R) = 0;$$

thus $H^{n-1}\{x \in A : |f(x)| = R\} = 0$ for almost every R.

Definition 3.1. Let $f: D \to \mathbb{R}^n$, $n \geqslant 2$, be a continuous mapping. For every $R > 0$ the mean inverse radius $I(R)$ is defined by

$$I(R) = I(R, f, D) = \left\{\frac{1}{\omega_{n-1}} H^{n-1}\{x \in D : |f(x)| = R\}\right\}^{1/(n-1)}. \tag{3.1}$$

THEOREM 3.1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $f: D \to \mathbb{R}^n$ be a K-quasiregular mapping. Then*

$$\int_0^\infty \frac{\omega_{n-1} I(R)^n}{p(R)^{1/(n-1)}} \cdot \frac{dR}{R} \leqslant K_I(f) m_n(D), \tag{3.2}$$

where

$$p(R) = p(R, f, D) = \frac{1}{\omega_{n-1} R^{n-1}} \left(\int_{S^{n-1}(R)} n(y, f, D) d\Lambda(y) \right), \tag{3.3}$$

$n(y, f, D)$ is as defined in (1.3) *and $d\Lambda$ is an element of spherical measure of $S^{n-1}(R)$. The integrand is taken to be zero if $p(R) = \infty$ or $I(R) = 0$; especially we obtain that $I(R) < \infty$ for almost every R for which $p(R) < \infty$.*

Proof. To begin with, we prove the theorem for two particular cases.

Case I, *in which the following hold.* The domain D is an n-dimensional cube; $f \neq 0$; fD lies in a cone C, where $C = \{x \in \mathbb{R}^n : (x, e) > |x|(\sqrt{2}/2), e \in S^{n-1}\}$; (x, y) is the inner product $\sum_{i=1}^n x_i y_i$; f is qc and has a homeomorphic extension to $\bar{D}$, which we denote again by f.

Let $h: \mathbb{R}^n \to \mathbb{R}^n$ be a rectifying diffeomorphism which sends sections $S^{n-1}(R) \cap fD = E'_R$ into $(n-1)$-planes parallel to $\{x \in \mathbb{R}^n : x_n = 0\}$. We can write h as the composition $h = h_2 \circ h_1$ where h_1 is a rotation of the cone C to the cone $\{(R, \phi, \theta_1, \dots, \theta_{n-2}) \in \mathbb{R}^n : \phi < \pi/4\}$ where $(R, \phi, \theta_1, \dots, \theta_{n-2})$ are spherical coordinates. The mapping h_2 is given by

$$h_2(R, \phi, \theta_1, \dots, \theta_{n-2}) = (R/\cos\phi, \phi, \theta_1, \dots, \theta_{n-2}).$$

Now from Agard [**1**; p. 44] we can show that for almost every R, $H^{n-1}\{f^{-1} \circ h^{-1}(F)\} < \infty$ for all compact $F \subset E''_R = h(E'_R)$. Write $E_R = f^{-1}\{E'_R\}$; then by Gehring [**2**; Lemma 6], for almost every R

$$H^{n-1}(E_R) = \int_{E_R} dH^{n-1} = \int_{E''_R} J(y, f^{-1} \circ h^{-1}|_{E''_R}) dm_{n-1}(y),$$

where

$$J(x,f) = \left(\sum_{i_1 < i_2 < \dots < i_{n-1}} \left(\frac{\partial(f_{i_1}, \dots, f_{i_{n-1}})}{\partial(x_1, \dots, x_{n-1})} \right)^2 \right)^{1/2}.$$

Furthermore, for almost every R the mapping $f^{-1} \circ h^{-1}$ is differentiable m_{n-1} a.e. in E''_R, as h is a diffeomorphism. Therefore

$$\omega_{n-1} I(R)^{n-1} = H^{n-1}\{E_R\} = \int_{E'_R} J\big(h(x), f^{-1} \circ h^{-1}|_{E''_R}\big) \cdot J(x, h|_{E'_R}) dH^{n-1}(x)$$

$$= \int_{E'_R} J(x, f^{-1}|_{E'_R}) dH^{n-1} \leqslant \int_{E'_R} |(f^{-1})'(x)|^{n-1} dH^{n-1},$$

which implies by Hölder's inequality, the definition of $p(R, f, D)$ and the quasiconformality of f

$$(\omega)_{n-1}^{n/(n-1)} I(R)^n = [H^{n-1}(E_R)]^{n/(n-1)} \leqslant \int_{E'_R} |(f^{-1})'(x)| dH^{n-1} \cdot \left(\int_{E'_R} dH^{n-1} \right)^{1/(n-1)}.$$

Applying Fubini's theorem and [**10**; 33.3], we find

$$\int_0^\infty \frac{\omega_{n-1} I(R)^n}{p(R)^{1/(n-1)}} \frac{dR}{R} = \int_0^\infty \frac{H^{n-1}(E_R)^{n/(n-1)}}{H^{n-1}(E'_R)^{1/(n-1)}} dR$$

$$\leqslant \int_{fD} |(f^{-1})'(x)|^n dm_n(x) \leqslant \int_{fD} K_O(f^{-1}) J(y, f^{-1}) dm_n$$

$$= K_I(f) m_n(D). \tag{3.4}$$

This completes the proof of Case I.

Case II, *when* $B_f = \emptyset$ *and* $f \neq 0$. We may assume that $m_n(D) < \infty$. We can divide the domain D by planes parallel to the coordinate axes into "half open" n-dimensional cubes D_ν, $\nu = 1, 2, \ldots$, such that $D = \bigcup \overline{D_\nu}$ and $D_\nu \cap \overline{D_\mu} = \emptyset$ for $\nu \neq \mu$. Moreover, since f is a local homeomorphism, we may assume that the division of D into cubes D_ν is sufficiently fine such that $f|_{\bar{D}_\nu}$ is one to one for every ν. We can also assume that each D_ν lies in a cone as described in Case I. Let D_ν be one of the covering cubes and let $(D_\nu)_i$, $i = 1, 2, \ldots, 2_n$, denote the $(n-1)$-dimensional faces of D_ν, and $M_{\nu,i}(R) = f^{-1}(S^{n-1}(R)) \cap (D_\nu)_i$. By Lemma 3.1, as $m_n((D_\nu)_i) = 0$, for almost every R we have $H^{n-1}\{M_{\nu,i}(R)\} = 0$. Thus, for almost every R

$$I(R)^{n-1} = \sum_{\nu=1}^{\infty} I(R, f, D_\nu)^{n-1} \tag{3.5}$$

and

$$p(R) = \sum_{\nu=1}^{\infty} p(R, f, D_\nu). \tag{3.6}$$

We have also

$$m_n(D) = \sum_{\nu=1}^{\infty} m_n(D_\nu). \tag{3.7}$$

Then Hölder's inequality, applied to the non-zero terms in (3.5–3.7) yields

$$\begin{aligned} I(R)^{n-1} &= \sum_{\nu=1}^{\infty} I(R, f, D_\nu)^{n-1} \\ &\leqslant \left\{ \sum_{\nu=1}^{\infty} \left[\frac{I(R, f, D_\nu)^n}{[p(R, f, D_\nu)]^{1/(n-1)}} \right] \right\}^{(n-1)/n} \cdot \left\{ \sum_{\nu=1}^{\infty} p(R, f, D_\nu) \right\}^{1/n}, \end{aligned}$$

and therefore

$$\frac{I(R, f, D)^n}{(p(R, f, D))^{1/(n-1)}} \leqslant \sum_{\nu=1}^{\infty} \frac{(I(R, f, D_\nu))^n}{(p(R, f, D_\nu))^{1/(n-1)}}.$$

Integrating from zero to ∞ and applying (3.4) to each of the cubes D_ν, we have

$$\begin{aligned} \int_0^\infty \frac{\omega_{n-1} I(R)^n}{R p(R)^{1/(n-1)}}\, dR &\leqslant \int_0^\infty \left(\sum_{\nu=1}^{\infty} \frac{\omega_{n-1}(I(R, f, D_\nu))^n}{(p(R, f, D_\nu))^{1/(n-1)}} \frac{dR}{R} \right) \\ &\leqslant \sum_{\nu=1}^{\infty} \int_0^\infty \frac{\omega_{n-1}(I(R, f, D_\nu))^n}{(p(R, f, D_\nu))^{1/(n-1)}} \frac{dR}{R} \\ &\leqslant K_I(f) \sum_{\nu=1}^{\infty} m_n(D_\nu) \leqslant K_I(f) m_n(D). \end{aligned}$$

This completes the proof of Case II.

We now turn to the general case. Let $D_0 = D\setminus B_f\setminus f^{-1}\{0\}$. Then the assertion of the theorem holds for $f_0 = f|_{D_0}$, by II. Since $m_n(f^{-1}(0)) = 0$ and $m_n(B_f) = 0$ [5], we have $m_n(D_0) = m_n(D)$. By Lemma 3.1

$$H^{n-1}\{f^{-1}(S^{n-1}(R)) \cap B_f\} = 0$$

for almost every R, and therefore it will follow that $I(R,f,D) = I(R,f,D_0)$. From the fact that $\Lambda(f(B_f) \cap S^{n-1}(R)) = 0$ it follows that $p(R,f,D_0) = p(R,f,D)$ for almost every R. With this (3.1) follows for a general f.

Remark (i). For plane analytic functions Theorem 3.1 reduces to the familiar theorem of Ahlfors and Cartwright [3; Theorem 2.1]. For plane qc mappings, this result is given in [9; Chapter XII].

Remark (ii). When this manuscript was completed, Prof. F. W. Gehring suggested a shorter proof for Theorem 3.1 based on the Co-Area formula in [11].

4. *Volume mean p-valent mapping*

Definition 4.1. Let $f: D \to \mathbb{R}^n$, $n \geqslant 2$, be a sense-preserving, discrete and open mapping; f is said to be *mean p-valent in volume* if

$$\omega(R) = \omega(R, D, f) = \int_0^R p(\rho)d(\rho^n) \leqslant pR^n, \tag{4.1}$$

where $p(\rho)$ is as defined in (3.3).

We can write

$$p(R) = p + h(R^n)$$

where p is a positive number. We can also write

$$\omega(R) = pR^n + \int_0^R h(\rho^n)d(\rho^n) = pR^n + H(R^n).$$

With this notation we can obtain the following Lemma.

LEMMA 4.1. *With the above notation*

$$\int_{R_1}^{R_2} \frac{d\rho}{\rho p(\rho)^{1/(n-1)}}$$

$$\geqslant \frac{1}{(np)^{1/(n-1)}}\left\{\log\frac{R_2}{R_1} - \frac{1}{n(n-1)} - \frac{H(R_2^n)}{n(n-1)pR_2^n} - \frac{1}{p(n-1)}\int_{R_1}^{R_2} \frac{H(\rho^n)}{\rho^{n+1}}d\rho\right\}.$$

Proof. It is clear that for every ρ, $h(\rho^n) \geqslant -p$ whenever $p(\rho) \geqslant 0$. If $-p < h(\rho^n) < 0$ we have

$$\frac{1}{\rho(p+h(\rho^n))^{1/(n-1)}} = \frac{1}{\rho p^{1/(n-1)}} \cdot \frac{1}{\left(1+\dfrac{h(\rho^n)}{p}\right)^{1/(n-1)}}$$

$$\geqslant \frac{1}{\rho p^{1/(n-1)}}\left(1-\frac{h(\rho^n)}{(n-1)p}\right) \geqslant \frac{1}{\rho(np)^{1/(n-1)}}\left(1-\frac{h(\rho^n)}{(n-1)p}\right)$$

since $(1+x)^{-1/(n-1)} \geqslant \left(1-\dfrac{x}{n-1}\right)$ for $-1 < x < 0$. If $h(\rho^n) \geqslant (n-1)p$ it is trivial that

$$\frac{1}{\rho(p+h(\rho^n))^{1/(n-1)}} \geqslant \frac{1}{\rho(np)^{1/(n-1)}}\left[1-\frac{h(\rho^n)}{(n-1)p}\right]. \tag{4.2}$$

If $0 \leqslant h(\rho^n) < (n-1)p$ then

$$\frac{1}{\rho(p+h(\rho^n))^{1/(n-1)}} \geqslant \frac{1}{\rho(np)^{1/(n-1)}} \geqslant \left(1-\frac{h(\rho^n)}{(n-1)p}\right)\frac{1}{\rho(np)^{1/(n-1)}}$$

and therefore (4.2) holds for every ρ. Integration by parts yields the lemma.

We would like to apply Theorem 3.1 to quasiregular mappings; a main tool for our application will be the following lemma.

LEMMA 4.2. *For $a > 0$ and $r > 0$ let G and $G(a)$ denote the following infinite and truncated cylinders, respectively, in cylindrical coordinates:*

$$G = \{(z, \rho, \theta_1, \ldots, \theta_{n-2}) : 0 \leqslant \rho < a\};$$

$$G_r(a) = \{(z, \rho, \theta_1, \ldots, \theta_{n-2}) : 0 \leqslant \rho < a,\ -a < z < r+a\}.$$

Let E_r denote the line segment $E_r = \{te_n : 0 \leqslant t \leqslant r\}$. Let $h : G \to \mathbb{R}^n \setminus \{0\}$ be a K-qr mapping with $|h(0)| = R_1$ and $|h(re_n)| = R_2$. Let $R_1 \leqslant R \leqslant R_2$.

(i) *If h is K-qr with $B_h = \varnothing$ and $n \geqslant 3$, then*

$$H^{n-1}\{x \in G : |h(x)| = R\} \geqslant H^{n-1}\{x \in G_r(a) : |h(x)| = R\} \geqslant C(n, K)a^{n-1} \tag{4.3}$$

where $C(n, K)$ is a positive constant depending on n and K.

(ii) *If h is K-qr and $n = 2$, then*

$$H^1\{x \in G : |h(x)| = R\} \geqslant 2a.$$

Proof. (i) Fix $R \in [R_1, R_2]$; h is qr and hence continuous, and since E_r is connected, there exists a point $\xi \in E_r$ such that $|h(\xi)| = R$.

If h is qr with $B_h = \varnothing$, and $n \geqslant 3$, it follows that h is injective in the ball $B^n(\xi, b)$ where $b = a \cdot \psi(n, K)$. Here, $\psi(n, K)$ is the universal radius of injectivity for the locally K-qc mapping in B^n, $n \geqslant 3$ (see [7]). Now we may apply [2; Theorem 2] to that ball and find

$$H^{n-1}\{x \in G : |h(x)| = R\} > H^{n-1}\{x \in B^n(\xi, b) : |h(x)| = R\}$$
$$\geqslant C(a \cdot \psi(n, K))^{n-1}$$

and (4.3) follows.

(ii) As in case (i) there exists a point $\xi \in E_r$ such that $|h(\xi)| = R$. There also exists one component of $\{x \in G : |h(x)| = R\}$ that passes through ξ. As $h : G \to \mathbb{R}^n \setminus \{0\}$ this component will go to the boundary of $G_r(a)$, and therefore (ii) is trivial.

Remark. It is reasonable to conjecture that (4.3) is true for general qr mappings, as it is not hard to prove that for general qr mappings $H^{n-1}\{x \in G : |h(x)| = R\} > 0$, but an argument of covering by balls where the mapping is qc does not work.

THEOREM 4.1. *Let $f: B^n \to \mathbb{R}^n \setminus \{0\}$ be a mean p-valent in volume K-qr mapping.*

(i) *If $n \geqslant 3$ and $B_f = \varnothing$, then*

$$|f(0)|\{e^{-2\sqrt{2}}(1-|x|)/(1+|x|)\}^{p^{1/(n-1)}c} < |f(x)| < |f(0)|\{e^{2\sqrt{2}}(1+|x|)/(1-|x|)\}^{p^{1/(n-1)}c}$$

where c is a positive constant depending on n and K.

(ii) *If $n = 2$ then*

$$|f(0)|\{e^{-\pi}(1-|x|)/(1+|x|)\}^{2pK_I(f)} < |f(x)| < |f(0)|\{e^{\pi}(1+|x|)/(1-|x|)\}^{2pK_I(f)}.$$

Proof. Write $R_1 = |f(0)|$ and $R_2 = |f(x)|$. (i) We assume that $R_1 < R_2$ (the argument for the other case is the same), and also that $x = re_n$. Let T be a Möbius transformation with $T(B^n) = H^+ = \{x \in \mathbb{R}^n : x_n > 0\}$, $T(0) = e_n$ and such that the line segment

$$E_r = \{x \in B^n : 0 \leqslant x_n < r,\ x_j = 0, j = 1, 2, \ldots, n-1\}$$

is mapped onto the line segment

$$T(E) = \{te_n : 1 \leqslant t \leqslant [(1+|x|)/(1-|x|)]\}.$$

We define a mapping g of H^+ onto the cylinder $G = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 < 1\right\}$ by

$$g(R, \phi, \theta_1, \ldots, \theta_{n-2}) = (z, r, \theta_1, \ldots, \theta_{n-2}),$$

$$z = \frac{1}{\sqrt{2}} \log R, \qquad r = \frac{1}{\sqrt{2}} \frac{\sin\phi}{\sin(\phi+\pi/4)}$$

where $z, r, \theta_1, \ldots, \theta_{n-2}$ are cylindrical coordinates in G, $0 \leqslant r < 1$; $0 \leqslant \theta_i < 2\pi$, $i = 1, 2, \ldots, n-2$ and where $R, \phi, \theta_1, \ldots, \theta_{n-2}$ are spherical coordinates in H^+. Here $0 \leqslant \phi < \pi/2$ is measured from e_n, and $0 \leqslant \theta_i < 2\pi$, $i = 1, 2, \ldots, n-2$. Then $g(T(0)) = 0$ and

$$g(T(re_n)) = \{(1/\sqrt{2}) \log [(1+|x|)/(1-|x|)] + 1\} e_n .$$

Then $h = f \circ T^{-1} \circ g^{-1}|_G$ is qr with $B_h = \varnothing$ if f is so; $|h(0)| = |f(0)| = R_1$ and

$$|h((1/\sqrt{2}) \log [(1+|x|)/(1-|x|)] e_n)| = |f(re_n)| = R_2 .$$

It is not hard to see that $K_I(g^{-1}) = \sqrt{2}$; then by Theorem 3.1 and Lemma 4.2, for $a = 1$ we obtain

$$\int_{R_1}^{R_2} \frac{c}{Rp(R)^{1/(n-1)}} \, dR \leqslant \int_{R_1}^{R_2} \frac{\omega_{n-1} I(R)^n}{Rp(R)^{1/(n-1)}} \, dR$$

$$\leqslant \sqrt{2} K_I(f) \Omega_{n-1} \{(1/\sqrt{2}) \log (1+|x|)/(1-|x|) + 2\} .$$

By Lemma 4.1 we obtain

$$\frac{1}{(np)^{1/(n-1)}} \left\{ \log \frac{R_2}{R_1} - \frac{1}{n(n-1)} \right\} \leqslant \frac{\sqrt{2} K_I(f) \Omega_{n-1}}{C} \{(1/\sqrt{2}) \log (e^{2\sqrt{2}}(1+|x|)/(1-|x|))\} .$$

Thus

$$\log \frac{R_2}{R_1} e^{-1/n(n-1)} \leqslant C(n, K) p^{1/(n-1)} \log \{e^{2\sqrt{2}}(1+|x|)/(1-|x|)\}$$

and the right-hand side of the inequality in (i) follows immediately. In order to find the lower bound in (i) and (ii), we take $h \circ f$ where h is a composition of two inversions, the first in the sphere S^{n-1} and the second in the plane $x_n = 0$. The composition $h \circ f$ is qr with $K_I(h \circ f) = K_I(f)$; as h is conformal, $h \circ f \colon B^n \to \mathbb{R}^n \setminus \{0\}$ satisfies the same conditions as f and thus

$$M(r, h \circ f) = \operatorname*{Max}_{|z| = r} |h \circ f(z)| = 1/m(r, f)$$

where $m(r,f) = \operatorname*{Min}_{|z| = r} |f(z)|$ and we get the lower bound from the upper bound.

For the case when $n = 2$ we use the argument which is used in the case of analytic functions (see [**3**; Theorem 2.2]), together with Theorem 3.1, and obtain our claim.

Using appropriate n-dimensional notation one can obtain an n-dimensional version of Lemmas 2.3 and 2.4 of [**3**]. With the help of these lemmas one can obtain the following theorem.

THEOREM 4.2. *Let $f: B^n \to \mathbb{R}^n$ be a K-qr mapping with $n(0,f,B^n) = q < \infty$. Then there exists $r_1 \in [0, 1/2]$ with $R_1 = \operatorname*{Min}_{|z|=r_1} |f(z)| > 0$, such that the following statements hold for every $r \in (r_1, 1)$.*

(i) *If $n \geqslant 3$ with $B_f = \varnothing$ then*

$$\int_{R_1}^{R_2} \frac{dR}{Rp(R)^{1/(n-1)}} < 2C\left\{\log\frac{1}{1-r}\right\} + B.$$

(ii) *If $n = 2$ then*

$$\int_{R_1}^{R_2} \frac{dR}{Rp(R)} < 2K_I(f)\left\{\log\frac{1}{1-r}\right\} + E.$$

Here $R_2 = \operatorname*{Max}_{|x|=r} |f(x)|$, C is a positive constant depending only on n and K, and B, E are positive constants depending on n, K, q and K, q respectively, and $q = n(0,f,B^n)$.

5. *Boundary behaviour of mean p-valent in volume quasiregular mappings with $B_f = \varnothing$*

Definition 5.1. Let $f: B^n \to \mathbb{R}^n$ be K-qr. Let $a \in S^{n-1}$; if there exists a path $\gamma: [0,1] \to \overline{B^n}$ such that $\gamma([0,1)) \subset B^n$, $\gamma(1) = a$, and a positive δ such that

$$\varliminf_{t\to 1} (1-|\gamma(t)|)^{\delta} |f(\gamma(t))| > 0,$$

then we define the lower order $\alpha = \alpha(a)$ of f at a as

$$\sup\left\{\delta > 0 : \varliminf_{t\to 1} (1-|\gamma(t)|)^{\delta}|f(\gamma(t))| > 0\right\}.$$

If no path γ and positive δ exist, we put $\alpha(a) = 0$.

Using Hölder's inequality instead of Schwarz's inequality one can obtain an n-dimensional version of Theorem 2.6 of [3], and therefore get the following n-dimensional version of Theorem 2.7 of [3].

THEOREM 5.1. *Let $f: B^n \to \mathbb{R}^n$ be K-qr and mean p-valent in volume ($p > 0$). Let E be the set defined as $E = \{x \in S^{n-1} : \alpha(x) > 0\}$. Then if $n = 2$ or $n \geqslant 3$ and $B_f = \varnothing$, the set E is countable and*

$$\sum_E \alpha(a)^{n-1} \leqslant p[2C]^{n-1}.$$

If $n = 2$, then $C = K_I(f)$, and for $n \geqslant 3$, C is a positive constant depending only on n and K.

Acknowledgments. This paper is a generalized form of a theorem and applications that were part of the author's D.Sc. Dissertation at the Technion Israel Institute of Technology. Part of an earlier version, appearing in [**8**], contains Theorem 3.1 in a weaker form, where

$$I(R) = \left\{\frac{1}{\omega_{n-1}} H^{n-1}\{x \in D \diagdown B_f : |f(x)| = R\}\right\}^{1/(n-1)}.$$

The applications that are mentioned are for spherically mean p-valent local homeomorphisms, and not mean p-valent in volume, a more general class.

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References

1. S. B. Agard, "Quasiconformal mappings and the moduli of p-dimensional surface families", *Proceedings of the Romanian–Finnish seminar on teichmüller spaces and quasiconformal mappings, Brasov, Romania*, 1969 (Publishing House of the Academy of RSR, Bucharest, 1971), pp. 9–48.
2. F. W. Gehring, "Lower dimensional absolute continuity properties of quasiconformal mappings", *Math. Proc. Cambridge Philos. Soc.*, 78 (1975), 81–93.
3. W. K. Hayman, *Multivalent functions* (Cambridge University Press, Cambridge, 1967).
4. O. Martio and S. Rickman, "Measure properties of the branch set and its image of quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A I*, 541 (1973).
5. O. Martio, S. Rickman and J. Väisälä, "Definitions for quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A I*, 448 (1969).
6. O. Martio, S. Rickman and J. Väisälä, "Distortion and singularities of quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A I*, 465 (1970).
7. O. Martio, S. Rickman and J. Väisälä, "Topological and metric properties of quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A I*, 488 (1971).
8. R. Miniowitz, "Quasiregular mappings", *Complex analysis, Joensuu* 1978 *Proceedings*, Lecture Notes in Mathematics 747 (Springer, Berlin, 1978), pp. 285–288.
9. R. Nevanlinna, *Analytic functions* (Springer, Berlin, 1970).
10. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics 229 (Springer, Berlin, 1971).
11. W. P. Ziemer, "Extremal length and conformal capacity", *Trans. Amer. Math. Soc.*, 126 (1967), 460–473.

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